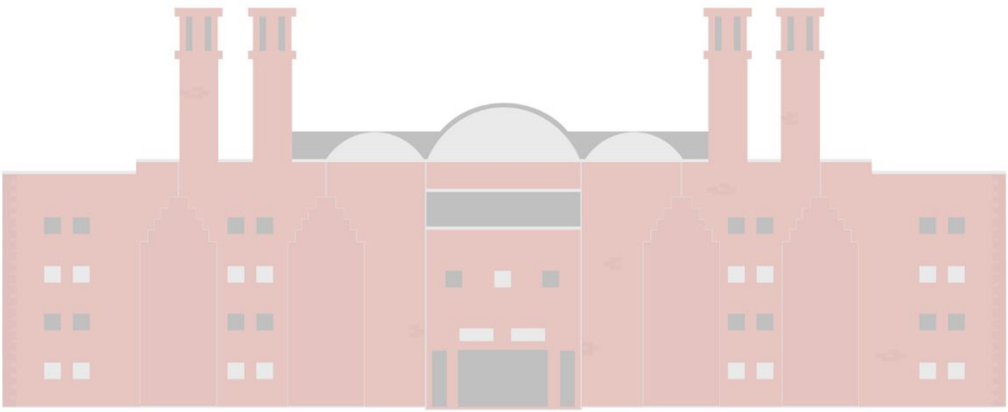
التحليل الرياضي ٣



الدكتور: معاذ عبد المجيد

التاريخ: ٢٠١٤/١٠/٢٦

٤-التوابع القطعية العكسية:

هي توابع تُنسب للتابع الأسي

$$\omega = \cosh^{-1} z = ch^{-1} z = arc\cosh z \ : \ \mathbb{C} \to \mathbb{C}$$

$$\Rightarrow z = \frac{1}{2}(e^{\omega} + e^{-\omega}) = e^{\omega} + e^{-\omega} = 2z \quad , \quad e^{\omega} = t$$

$$\Rightarrow t + \frac{1}{t} = 2z \quad \Rightarrow t^2 + 1 - 2zt = 0$$

$$\Delta = (-2z)^2 - 4 = 4z^2 - 4$$

$$\Rightarrow t = \frac{2z \mp \sqrt{4z^2 - 4}}{2} = z \mp \sqrt{z^2 - 1}$$

$$\omega = \cosh^{-1} z = \ln\left(z \mp \sqrt{z^2 - 1}\right)$$

$z^2 - 1 = 0 \ \Rightarrow (z = \mp 1)$ نقطتي التفرع

$$\omega = \sinh z = \ln\left(z \mp \sqrt{z^2 - 1}\right) \quad : \quad \mathbb{C} \to \mathbb{C}$$

$(z = \mp i)$ نقطتي التفرع

$$\omega = \tanh^{-1} z = th^{-1} z = \frac{1}{2i} \ln\left(\frac{1+z}{1-z}\right)$$

وظيفة :

حل المعادلات التالية :

$$ch\, z = i \,, \sin z \sqrt{10} \quad , Re(z^z), \sinh^{-1}[\ln(-1)]$$

$$\left\|(-i)^{-i}\right\| , 1^{\sqrt{2}} , R_e\left[(-i)^{-i}\right]$$

مثال :

$$i^i = e^{\cosh^{-1} z}$$

$$e^{i\left(\frac{\pi}{2} + 2\pi k\right)i} = e^{\ln\left(z + \sqrt{z^2 - 1}\right)} = \left(z + \sqrt{z^2 - 1}\right)$$

$$\sqrt{z^2 - 1} = -z + e^{-\frac{\pi}{2} - 2\pi k}$$

$$z^2 - 1 = z^2 - 2ze^{-\frac{\pi}{2} - 2\pi k} + e^{-\pi - 4\pi k}$$

$$2ze^{-\frac{\pi}{2} - 2\pi k} = 1 + e^{-\pi - 4\pi k}$$

$$z = \frac{1 + e^{-\pi - 4\pi k}}{2e^{-\frac{\pi}{2} - 2\pi k}}$$

$$z = \frac{1}{2}\left[e^{\frac{\pi}{2} + 2\pi k} + e^{-\frac{\pi}{2} - 2\pi k}\right]$$

$$\left(z = \cosh\left(\frac{\pi}{2} + 2\pi k\right)\right)$$

مثال: بيّن أنّ هذا التابع توافقي و أوجد مرافقه التوافقي بحيث يكون $f(z)$ تحليلي و أوجد $f(z)$.

$$v = xy$$
$$\nabla^2 v = v''_{xx} + v''_{yy} = 0$$
$$v'_x = y \to v''_{xx} = 0$$
$$v'_y = x \to v''_{yy} = 0$$
$$\nabla^2 v = 0 \Rightarrow \text{توافقي } v$$

$$u'_x = v'_y$$
$$\Rightarrow u = \int v'_y\,dx + h(y) = \int x\,dx + h(y)$$
$$\left(u = \frac{1}{2}x^2 + h(y)\right) \ldots\ldots\ldots\ldots\ldots \mathbf{I}$$
$$u'_y = -v'_x$$
$$h'(y) = -y \Rightarrow h(y) = -\frac{1}{2}y^2 + c$$
$$\left(u = \frac{1}{2}x^2 - \frac{1}{2}y^2 + c\right) \ldots\ldots\ldots\ldots\ldots \mathbf{II}$$
$$f(z) = u + iv$$
$$f(z) = \left(\frac{1}{2}x^2 - \frac{1}{2}y^2 + c\right) + ixy$$

$$\left(f(z) = \frac{1}{2}z^2 + c\right)$$

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$$u = \ln|z| = \frac{1}{2}\ln(x^2 + y^2)$$
$$u = \ln\rho$$

$$\nabla^2 u == u''_{\rho\rho} + \frac{1}{\rho}u'_\rho + \frac{1}{\rho^2}u''_{\varphi\varphi} = 0$$
$$u'_\rho = \frac{1}{\rho} \Rightarrow u''_{\rho^2} = -\frac{1}{\rho^2}\,, u''_{\varphi^2} = 0$$
$$\nabla^2 u = -\frac{1}{\rho^2} + \frac{1}{\rho} - \frac{1}{\rho} + 0 = 0 \Rightarrow \text{توافقي } u$$
$$\left(u'_\rho = \frac{1}{\rho}v'_\varphi\right) \ldots\ldots\ldots\ldots \mathbf{I}$$

$$\frac{1}{\rho} = \frac{1}{\rho} v'_{\varphi} \Rightarrow v'_{\varphi} = 1$$

$$v(\rho,\varphi) = \int 1\, d\varphi + h(\rho)$$

$$v(\rho,\varphi) = \varphi + h(\rho)$$

$$u'_{\varphi} = -\rho v' \ \ldots\ldots\ldots \mathbf{II}$$

$$v'_{\rho} = h'(\rho)$$

$$u'_{\varphi} = 0 \Rightarrow -\rho h'(\rho) = 0$$

$$\Rightarrow h'(\rho) = 0 \Rightarrow h(\rho) = c$$

$$u = \varphi + c$$

$$f(z) = u + iv = \ln \rho + i(\varphi + c) = \ln z + ic$$

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$$u = ax^3 + by^3$$

$$\nabla^2 u = u''_{x^2} + u''_{y^2} = 0$$

$$u''_{x^2} = 6ax \ , u''_{y^2} = 6by$$

$$ax + by = 0 \Rightarrow a = y\, , b = -x$$

$$u = x^3y - y^3x$$

$$u'_x = 3x^2y - y^3$$

$$u'_x = +v'_y \ \ldots\ldots\ldots \mathbf{I}$$

$$v = \int u'_x\, dy + h(x)$$

$$\left(v = \frac{3}{2}x^2y^2 - \frac{1}{4}y^4 + h(x)\right)$$

$$u'_y = -v'_x$$

$$x^3 - 3xy^2 = -\left(3xy^2 + h'(x)\right)$$

$$\left(h'(x) = -x^3\right)$$

$$h(x) = -\frac{1}{4}x^4 + c$$

$$v = \frac{2}{3}x^2y^2 - \frac{1}{4}y^4 - \frac{1}{4}x^4 + c$$

$$f(z) = 0 + i\left(-\frac{1}{4}z^4 + c\right)$$

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$$u = e^{3x}\cos ay$$

$$u'_x = 3e^{3x}\cos ay \quad , \quad u''_{x^2} = 9e^{3x}\cos ay$$

$$u'_y = -ae^{3x}\sin ay \quad , \quad u''_{y^2} = -a^2e^{3x}\cos ay$$

$$\nabla^2 u = 9e^{3x}\cos ay - a^2e^{3x}\cos ay = 0$$
$$a^2 = 9 \Rightarrow a = \mp 3$$

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$$u = ax^2y + by^2 - 3y^3 + 2x^2$$
$$u'_x = 2axy + 4x$$
$$u''_{x^2} = 2ay + 4$$
$$u'_y = 2by - 9y^2 + ax^2$$
$$u''_{y^2} = 2b - 18y$$

$$\nabla^2 u = 2ay + 4 + 2b - 18y = 0$$
$$4 + 2b = 0 \ , 2a - 18 = 0$$
$$\Rightarrow b = -2 \ , a = 9$$
$$u = 9x^2y - 2y^2 - 3y^3 + 2x^2$$

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بيّن أنّ هذا التابع تحليلي:

$$f(z) = \mathrm{Im}(z^2) = \mathrm{Im}(x^2 - y^2 + 2xyi) = 2xy$$
$$f(z) = 2xy + 0i$$
$$u'_x = 2y = v'_y = 0 \Rightarrow y = 0$$
$$u'_y = 2x = -v'_x = 0 \Rightarrow x = 0$$

f تحليلي في $\mathbb{C}$ باستثناء المحاور الإحداثية .

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$$e^{2z} = 2 - i = \sqrt{5}\, e^{i\theta}$$
$$\cos\theta = \frac{2}{\sqrt{5}}\ , \sin\theta = -\frac{1}{\sqrt{5}}$$
$$2z = \ln\left(\sqrt{5}\, e^{i\theta}\right) = \ln\sqrt{5} + i(\theta + 2\pi k)$$
$$z = \frac{1}{4}\ln 5 + i\left(\frac{\theta}{2} + \pi k\right) \ : \ \ k \in Z$$

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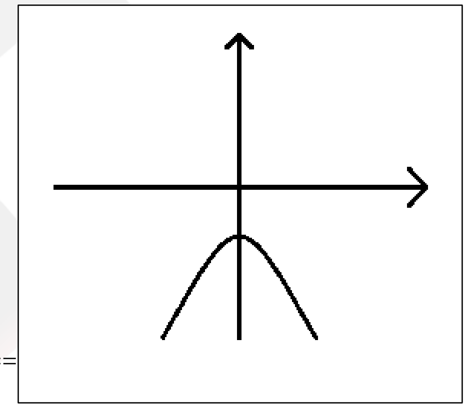
ماذا تمثل هذه المعادلة و ارسم المنطقة :

$$\operatorname{Im}(z) \geq \|z + 2i\| \geq 0$$
$$y \geq \sqrt{x^2 + (y+2)^2} \geq 0$$
$$y^2 \geq x^2 + y^2 + 4y + 4 \geq 0$$
$$0 \geq x^2 + 4y + 4 \geq 0$$
$$\Rightarrow x^2 + 4y + 4 = 0$$



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$$Re\left[(-i)^{-i}\right]$$
$$(-i)^{-i} = e^{\left(i + \frac{3\pi}{2} + 2\pi k\right)(-i)}$$
$$(-i)^{-i} = e^{\frac{3\pi}{2} + 2\pi k}$$
$$Re(-i)^{-i} = e^{\frac{3\pi}{2} + 2\pi k} \quad : k \neq 0$$

إن الأشخاص الهادئين يمتلكون أكثر العقول

" صخباً "

ستيفن هوكينج